Nonlocal Problem with Bitsadze-Samarsky and Samarsky-Ionkin Type Conditions for a System of Pseudoparabolic Equations

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***Abstract*— In the paper in S.L.Sobolev anisotropic space a Bitsadze-Samarsky and Samarsky-Ionkin-type non-local combined problem is investigated for a system of fifth order pseudoparabolic equations with non-smooth coefficients.**

Keywords*— *Bitsadze-Samarsky type problems, Samarsky-Ionkin type prob-lem, discontinouns coefficients equations.

I. INTRODUCTION

Pseudoparabolic equations are used for rather adequate description of a great majority of real processes occurring in nature, engineering and etc. A lot of processes happening in theory of fluid filtration in fissued media are described by pseudoparabolic equations with non smooth coefficients. Therewith the mathematical model of the process is completed by Bitsadze-Samarsky and Samarsky-Ionkin type non local boundary conditions. Therefore the themes of the present paper is very urgent for solving such theoretical and practical problems.

II. PROBLEM STATEMENT

Given the equation

$$(V_{2,3}u)(t,x) \equiv \sum_{i=0}^{2}\sum_{j=0}^{3}\left(D_t^i D_x^j u(t,x)\right)A_{i,j}(t,x) = \\ = Z_{2,3}(t,x),\ (t,x)\in G=(t_0,t_1)\times(x_0,x_1) \tag{1}$$

with the initial conditions

$$\begin{cases} (l_0u)(x) \equiv u(t_0,x) = Z_0(x),\ x\in(x_0,x_1) \\ (l_1u)(x) \equiv u_t(t_0,x) = Z_1(x),\ x\in(x_0,x_1) \end{cases} \tag{2}$$

and Bitsadze-Samarsky and Samarsky-Ionkin type boundary conditions [1-3]:

$$\begin{aligned} &(l_2^0u)(t) \equiv u(t,x_0)\alpha_{1,1} + u_x(t,x_0)\alpha_{1,2} + u_{xx}(t,x_0)\alpha_{1,3} + u(t,x_1)\beta_{1,1} + \\ &+u_x(t,x_1)\beta_{1,2} + u_{xx}(t,x_1)\beta_{1,3} = \psi_2(t), t\in(t_0,t_1); \\ &(l_3^0u)(t) \equiv u(t,x_0)\alpha_{2,1} + u_x(t,x_0)\alpha_{2,2} + u_{xx}(t,x_0)\alpha_{2,3} + u(t,x_1)\beta_{2,1} + \\ &+u_x(t,x_1)\beta_{2,2} + u_{xx}(t,x_1)\beta_{2,3} = \psi_3(t),\ t\in(t_0,t_1); \\ &(l_4^0u)(t) \equiv u(t,x_0)\alpha_{3,1} + u_x(t,x_0)\alpha_{3,2} + u_{xx}(t,x_0)\alpha_{3,3} + u(t,x_1)\beta_{3,1} + \\ &+u_x(t,x_1)\beta_{3,2} + u_{xx}(t,x_1)\beta_{3,3} = \psi_4(t), t\in(t_0,t_1). \end{aligned} \tag{3}$$

Here: $u(t,x)=(u_1(t,x),...,u_n(t,x))$ is an n-dimensional desired vector function; $\alpha_{i,j}$ and $\beta_{i,j}$ are the given $n\times n$-dimensional constant matrices, $A_{i,j}(t,x)$ are measurable on G matrix functions of order $n\times n$ and satisfying the conditions:

$A_{i,j}(t,x)\in L_p(G),\ i=\overline{0,1},\ j=\overline{0,2}$, and there exist the functions $A_{2,j}^0(x)\in L_p(x_0,x_1)$ and $A_{i,3}^0(t)\in L_p(t_0,t_1)$ such that the conditions

$$\|A_{2,j}(t,x)\| \le A_{2,j}^0(x),\ j=\overline{0,2} \text{ и } \|A_{i,3}(t,x)\| \le A_{i,3}^0(t), i=\overline{0,1}$$

are fulfilled almost everywhere on G, where $\|\ \|$ is an Euclidean norm of the corresponding matrix (or vector);

$$A_{2,3}(t,x)\equiv E;\ Z_0(x)\in W_{p,n}^{(3)}(x_0,x_1),\ Z_1(x)\in W_{p,n}^{(3)}(x_0,x_1),$$

and also $\psi_2(t),\psi_3(t),\psi_4(t)\in W_{p,n}^{(2)}(t_0,t_1)$ are the given n-dimensional vector - functions, where $W_{p,n}^{(m)}(y_0,y_1)$ is a space of n - dimensional vector-matrices $Z(y)=(Z_1(y),...,Z_n(y))$, having in S.L.Sobolev sense the derivatives

$$Z'(y),...,Z^{(m)}(y)\in L_{p,n}(y_0,y_1),$$

and $L_{p,n}(y_0,y_1)$ is a space of all line vectors $Z(y)=(Z_1(y),...,Z_n(y))$ with the elements from $Z_i(y)\in L_p(y_0,y_1)$, $i=1,...,n$. We'll look for the solution of problem (1)-(3) in S.L.Sobolev space

$$W_{p,n}^{(2,3)}(G)=\left\{u\in L_{p,n}(G)\setminus D_t^iD_x^ju\in L_{p,n}(G), i=\overline{0,2},\ j=\overline{0,3}\right\},$$

where, $1 \le p \le \infty$. Define the norm in this space by-the equality

$$\|u\|_{W_{p,n}^{(2,3)}} = \sum_{i=0}^{2}\sum_{j=0}^{3}\left\|D_t^i D_x^j u\right\|_{L_{p,n}(G)}.$$

Obviously, the right sides $Z_0(x), Z_1(x)$ and $\psi_i(t), i=\overline{2,4}$ of conditions (2) and (3) should satisfy the agreement conditions

$$\begin{aligned}
&Z_0(x_0)\alpha_{1,1} + Z_0'(x_0)\alpha_{1,2} + Z_0''(x_0)\alpha_{1,3} + Z_0(x_1)\beta_{1,1} + \\
&+ Z_0'(x_1)\beta_{1,2} + Z_0''(x_1)\beta_{1,3} = \psi_2(t_0);\\
&Z_0(x_0)\alpha_{2,1} + Z_0'(x_0)\alpha_{2,2} + Z_0''(x_0)\alpha_{2,3} + Z_0(x_1)\beta_{2,1} + \\
&+ Z_0'(x_1)\beta_{2,2} + Z_0''(x_1)\beta_{2,3} = \psi_3(t_0);\\
&Z_0(x_0)\alpha_{3,1} + Z_0'(x_0)\alpha_{3,2} + Z_0''(x_0)\alpha_{3,3} + Z_0(x_1)\beta_{3,1} + \\
&+ Z_0'(x_1)\beta_{3,2} + Z_0''(x_1)\beta_{3,3} = \psi_4(t_0);\\
&Z_1(x_0)\alpha_{1,1} + Z_1'(x_0)\alpha_{1,2} + Z_1''(x_0)\alpha_{1,3} + Z_1(x_1)\beta_{1,1} + \\
&+ Z_1'(x_1)\beta_{1,2} + Z_1''(x_1)\beta_{1,3} = \psi_2'(t_0);\\
&Z_1(x_0)\alpha_{2,1} + Z_1'(x_0)\alpha_{2,2} + Z_1''(x_0)\alpha_{2,3} + Z_1(x_1)\beta_{2,1} + \\
&+ Z_1'(x_1)\beta_{2,2} + Z_1''(x_1)\beta_{2,3} = \psi_3'(t_0);\\
&Z_1(x_0)\alpha_{3,1} + Z_1'(x_0)\alpha_{3,2} + Z_1''(x_0)\alpha_{3,3} + Z_1(x_1)\beta_{3,1} + \\
&+ Z_1'(x_1)\beta_{3,2} + Z_1''(x_1)\beta_{3,3} = \psi_4'(t_0);
\end{aligned} \tag{4}$$

Availability of the agreement conditions means that some unnecessary informations on the solution were given by conditions (2) and (3). Therefore, it is desirable there would be no necessity in agreement-type conditions in the problem statement.

For obtaining such conditions, we'll twice differentiate condition (3) with respect to t. Then we get

$$\begin{cases}
(l_2u)(t) \equiv u_{tt}(t,x_0)\alpha_{1,1} + u_{ttx}(t,x_0)\alpha_{1,2} + u_{ttxx}(t,x_0)\alpha_{1,3} + u_{tt}(t,x_1)\beta_{1,1} + \\
+ u_{ttx}(t,x_1)\beta_{1,2} + u_{ttxx}(t,x_1)\beta_{1,3} = Z_2(t) = \psi_2''(t);\\
(l_3u)(t) \equiv u_{tt}(t,x_0)\alpha_{2,1} + u_{ttx}(t,x_0)\alpha_{2,2} + u_{ttxx}(t,x_0)\alpha_{2,3} + u_{tt}(t,x_1)\beta_{2,1} + \\
+ u_{ttx}(t,x_1)\beta_{2,2} + u_{ttxx}(t,x_1)\beta_{2,3} = Z_3(t) = \psi_3''(t);\\
(l_4u)(t) \equiv u_{tt}(t,x_0)\alpha_{3,1} + u_{ttx}(t,x_0)\alpha_{3,2} + u_{ttxx}(t,x_0)\alpha_{3,3} + u_{tt}(t,x_1)\beta_{3,1} + \\
+ u_{ttx}(t,x_1)\beta_{3,2} + u_{ttxx}(t,x_1)\beta_{3,3} = Z_4(t) = \psi_4''(t).
\end{cases} \tag{5}$$

Here we'll require that the conditions

$$Z_i(t) \in L_{p,n}(t_0,t_1), i=\overline{2,4}$$

be fulfilled.

It is obvious that if $u \in W_{p,n}^{(2,3)}(G)$ is a solution of problem (1), (2), (5),then it is also a solution of problem (1)-(3) for

$$\begin{aligned}
\psi_2(t) = &\int_{t_0}^{t}(t-\tau)Z_2(\tau)d\tau + Z_0(x_0)\alpha_{1,1} + Z_0'(x_0)\alpha_{1,2} \\
&+ Z_0''(x_0)\alpha_{1,3} + Z_0(x_1)\beta_{1,1} + Z_0'(x_1)\beta_{1,2} + Z_0''(x_1)\beta_{1,3} + \\
&+ (t-t_0)\big(Z_1(x_0)\alpha_{1,1} + Z_1'(x_0)\alpha_{1,2} + Z_1''(x_0)\alpha_{1,3} + \\
&+ Z_1(x_1)\beta_{1,1} + Z_1'(x_1)\beta_{1,2} + Z_1''(x_1)\beta_{1,3}\big);
\end{aligned}$$

$$\begin{aligned}
\psi_3(t) = &\int_{t_0}^{t}(t-\tau)Z_3(\tau)d\tau + Z_0(x_0)\alpha_{2,1} + Z_0'(x_0)\alpha_{2,2} + \\
&+ Z_0''(x_0)\alpha_{2,3} + Z_0(x_1)\beta_{2,1} + Z_0'(x_1)\beta_{2,2} + Z_0''(x_1)\beta_{2,3} + \\
&+ (t-t_0)\big(Z_1(x_0)\alpha_{2,1} + Z_1'(x_0)\alpha_{2,2} + Z_1''(x_0)\alpha_{2,3} + \\
&+ Z_1(x_1)\beta_{2,1} + Z_1'(x_1)\beta_{2,2} + Z_1''(x_1)\beta_{2,3}\big);
\end{aligned}$$

$$\begin{aligned}
\psi_4(t) = &\int_{t_0}^{t}(t-\tau)Z_4(\tau)d\tau + Z_0(x_0)\alpha_{3,1} + Z_0'(x_0)\alpha_{3,2} + \\
&+ Z_0''(x_0)\alpha_{3,3} + Z_0(x_1)\beta_{3,1} + Z_0'(x_1)\beta_{3,2} + Z_0''(x_1)\beta_{3,3} + \\
&+ (t-t_0)\big(Z_1(x_0)\alpha_{3,1} + Z_1'(x_0)\alpha_{3,2} + Z_1''(x_0)\alpha_{3,3} + \\
&+ Z_1(x_1)\beta_{3,1} + Z_1'(x_1)\beta_{3,2} + Z_1''(x_1)\beta_{3,3}\big).
\end{aligned} \tag{6}$$

Note that for the functions (6) the agreement conditions (4) are trivially fulfilled. The inverse is also, true i.e. if $u \in W_{p,n}^{(2,3)}(G)$ is a solution of problem (1)-(3), then it is also a solution of problem (1), (2), (5) for $Z_2(t) = \psi_2''(t), Z_3(t) = \psi_3''(t)$ and $Z_4(t) = \psi_4''(t)$. In other words, (1)-(3) and (1), (2), (5) are equivalent in $W_{p,n}^{(2,3)}(G)$. Therewith the solution of problem (1), (2), (5) is the solution of problem (1)-(3) for some $\psi_2(t), \psi_3(t), \psi_4(t)$ satisfying the agreement conditions automatically.

However, by the statement, problem (1), (2), (5) is more natural than (1)-(3). Therefore, in future we'll research only problem (1), (2), (5).

It should be especially noted that the Bitsadze -Samarsky and Samarsky-Ionkin type problems are investigated beginning with the paper [4] and were developed in [5-8] and others. Furthermore, note that such Bitsadze-Samarsky and Samarsky-Ionkin type non local boundary value problems in modified treatments were considered in the author's papers [9-10].

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